Stochastic and deterministic parametrizations for 2D-turbulence

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Model equations

- Incompressible fluid on domain $[0,2\pi) \times [0,2\pi)$ with periodic b.c.
- Biharmonic damping $-\mu\Delta^2\omega$
- Raleigh friction $-\alpha\omega$
- Stochastic forcing f of fixed spatial scale with wavenumber $k_f = 90$

$$\frac{\partial\omega}{\partial t} + J(\psi,\omega) = -\mu\Delta^2\omega - \alpha\omega + f$$
$$\Delta\psi = \omega$$

where ψ – stream function, ω – vorticity

Numerical schemes

- **E**, skew-symmetric energy-conserving scheme

$$[(\boldsymbol{u}\cdot\nabla)u_i]_h = \frac{1}{2}\delta_{x_j}\left(\overline{u_j}^{x_i}\overline{u_i}^{x_j}\right) + \frac{1}{2}\overline{\overline{u_j}^{x_i}\delta_{x_j}u_i}^{x_j}$$

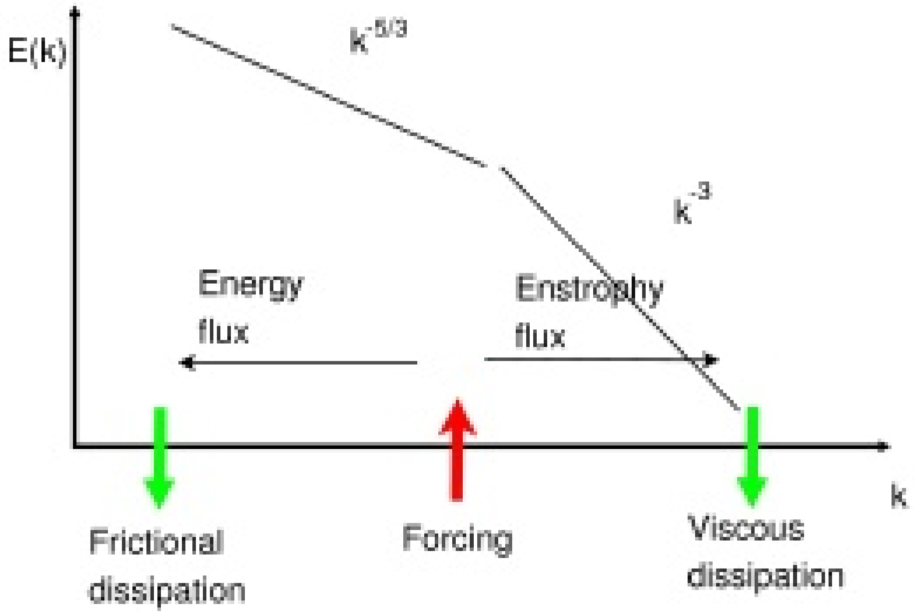
- **INMCM**, one of Arakawa schemes (Arakawa, 1977)

$$[(\boldsymbol{u}\cdot\nabla)u_i]_h = \frac{2}{3}\overline{\overline{u_j}^{x_i}\delta_{x_j}u_i}^{x_j} + \frac{1}{3}\overline{u_j'\delta_{x_j'}u_i}^{x_j'}$$

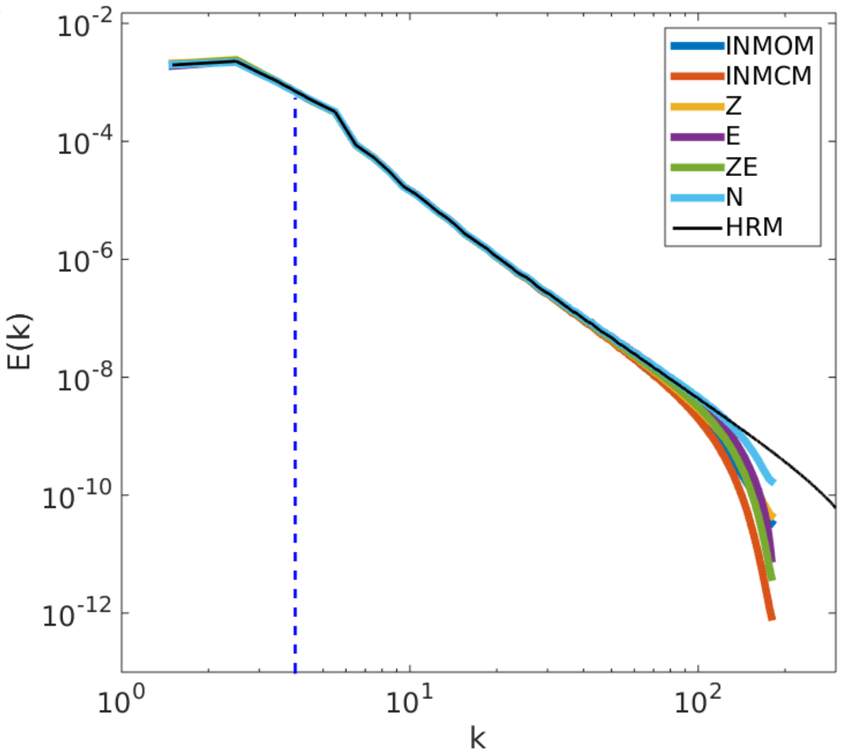
- **Z**, skew-symmetric enstrophy-conserving scheme (Arakawa, 1966)
- **CCS**, finite volume Semi-Lagrangian scheme (Nair, 2002)

Theory KLB(Kraichnan-Leith-Batchelor)

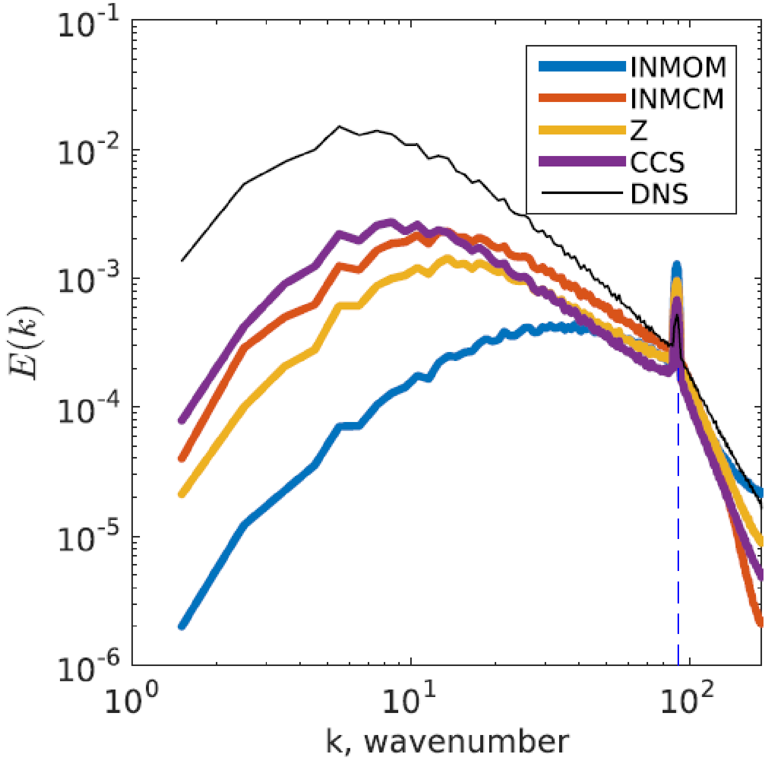
- Enstrophy ($Z = \frac{1}{2}\int \omega^2 dx$) moves to small scales
- Energy ($E = \frac{1}{2}\int u^2 dx$) moves to large scales



Summary of previous work



a) large scale forcing



b) small scale forcing

- Importance of numerical schemes properties depends on resolution
- All coarse models fail in the case of small scale forcing

A priori analysis of subgrid forces

Dynamics of **coarse model** is represented by filtered equations (spectral filtration denoted by overline):

$$\frac{\partial \overline{\omega}}{\partial t} + J^h(\bar{\psi}, \overline{\omega}) = \cdots + \sigma$$

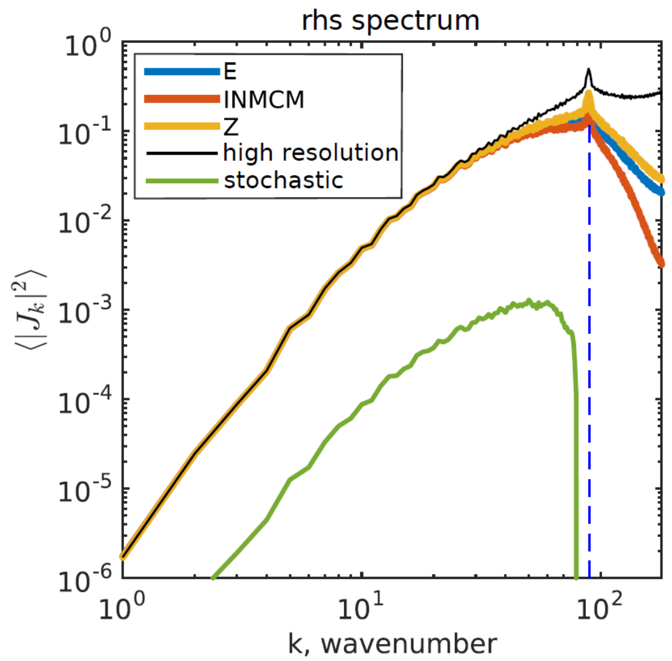
where σ – **subgrid forces** accounting for unresolved scales and numerical approximation $J^h(*,*)$:

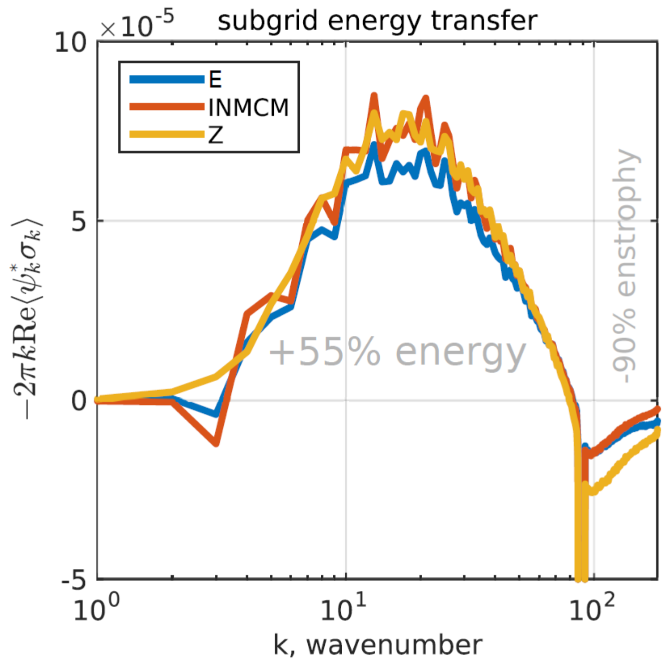
$$\sigma = J^h(\bar{\psi}, \overline{\omega}) - \overline{J(\psi, \omega)}$$

We run **high resolution** model 2160×2160 and gather statistics of subgrid forces for coarse models 360×360.

Forcing scale is 4 mesh steps of coarse model ($k_f = 90,\ k_{max} = 180$).

Spectral properties of subgrid forces





- rhs spectrum of advection in large scales is well represented by all coarse models
- subgrid energy generation is comparable with forcing power and injects energy into the large scales (**backscatter**)

On short time intervals coarse models reproduce large-scale variability well. However during long time integration the absence of backscatter parametrization leads to the **slow decay of large scale flows**, and inverse energy cascade eventually breaks.

Backscatter parametrizations - 1

stochastic

Ornstein-Uhlenbeck stochastic process in Fourier space (Berner, 2009). *Decorrelation time* and *energy generation* of subgrid forces are simulated by adjusting constants β_k and γ_k.

$$\frac{\partial \omega_k}{\partial t} = \dots + s_k$$

$$\frac{\partial s_k}{\partial t} = -\beta_k s_k + \gamma_k \varepsilon_k(t)$$

where $\varepsilon_k(t)$ – white noise with unit variance

Backscatter parametrizations - 2

eddy viscosity

Linear model in Fourier space (Kraichnan 1976). *Energy generation* of subgrid forces is simulated by adjusting negative coefficient $\nu(k)$.

$$\frac{\partial \omega_k}{\partial t} = \cdots - \nu(k) k^2 \omega_k, \nu(k) < 0$$

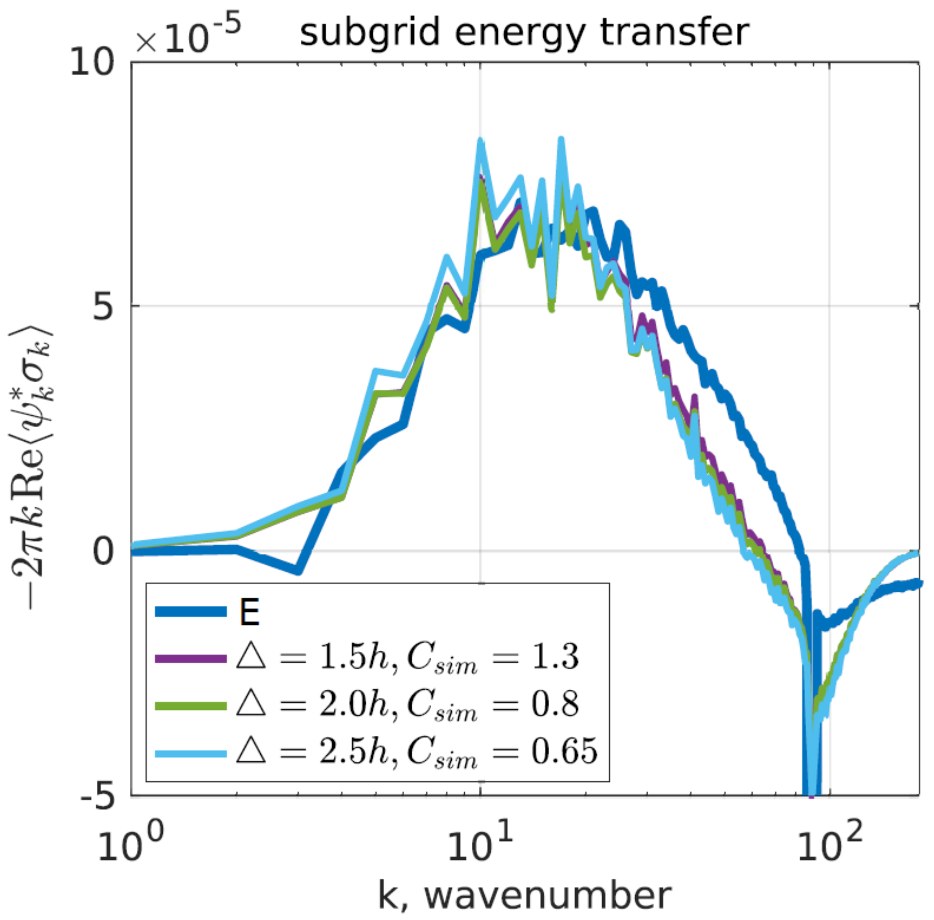
A priori analysis of scale similarity model

$$\frac{\partial \omega}{\partial t} = \dots + c_{sim}\frac{\partial \widetilde{l_j}}{\partial x_j}$$

$$l_j = \widehat{u}_j\widehat{\omega} - \widehat{u_j\omega}$$

Here $\widehat{(\cdot)}$ – test filter of width twice the mesh step, $\widetilde{(\cdot)}$ – additional spectral filter that remove scales smaller then forcing scale.

- Scale similarity model reproduce shape of energy backscatter spectral distribution



Backscatter parametrizations - 3

stochastic+similarity

Combined model incorporating stochastic and deterministic parts. Constants c_{stoch} and c_{sim} are adjusted in series of preliminary experiments with coarse model and chosen to fully compensate energy loss due to viscosity and scheme dissipation. Also, distribution of energy generation between stochastic and deterministic parts implemented in such a way as to get best results in large and middle scales at the same time.

$$\frac{\partial \omega}{\partial t} = \cdots + c_{stoch} s + c_{sim} \frac{\partial \widetilde{l_j}}{\partial x_j}$$

$$l_j = \widehat{u_j} \widehat{\omega} - \widehat{u_j \omega}$$

Here $\widehat{(\cdot)}$ – test filter of width twice the mesh step, $\widetilde{(\cdot)}$ – additional spectral filter that remove scales smaller then forcing scale.

Experiments with coarse models - 1

- Stochastic and eddy viscosity models effectively restore large scales
- Scale-similarity model restores middle scales (not shown)
- Combined model gives the best result: full inertial range of energy cascade was restored

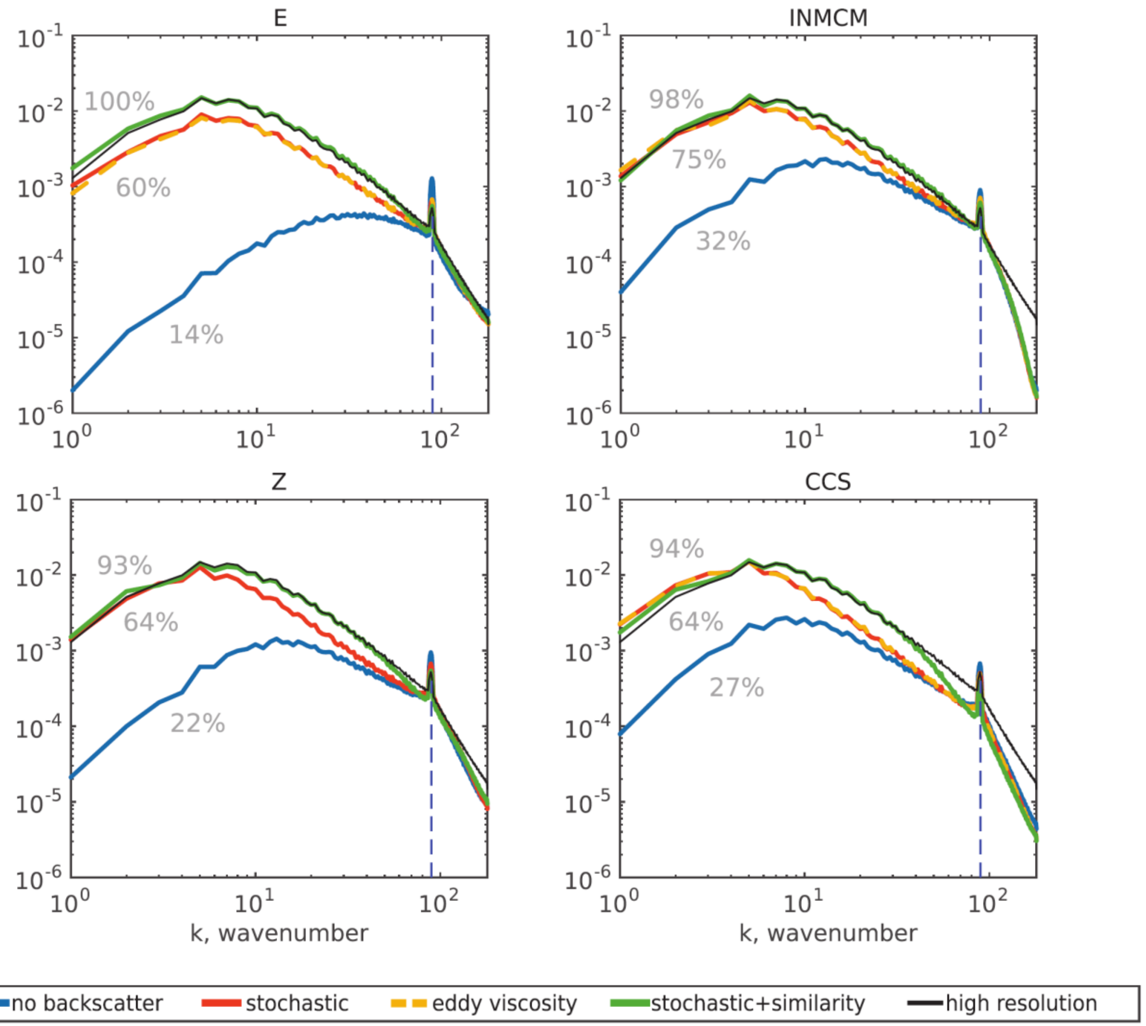


Fig. 2. Energy spectrum for different schemes (E, INMCM, Z, CCS).

Experiments with coarse models - 2

- Large scale flows emerged

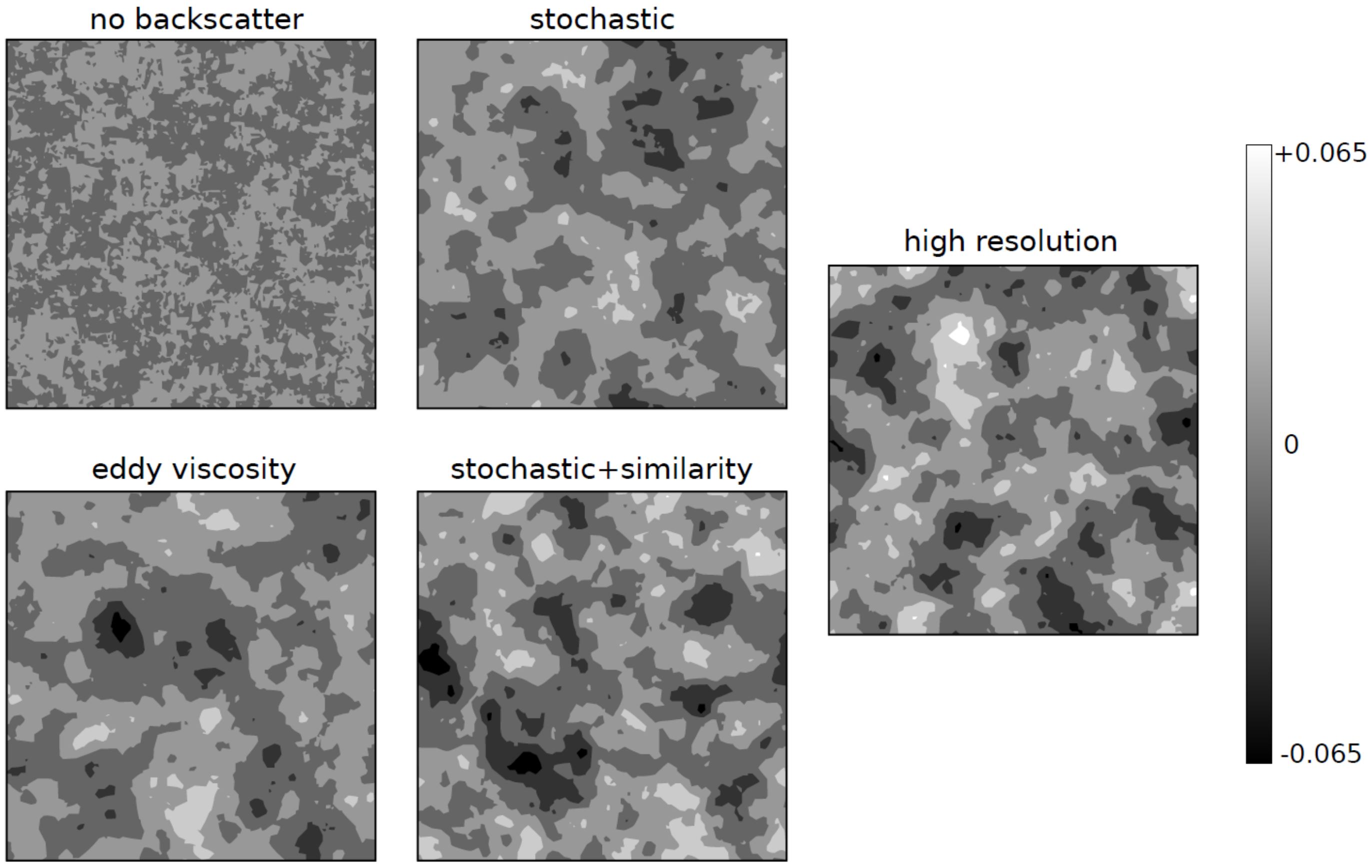


Fig. 3. Stream function patterns for scheme E.

Experiments with coarse models - 3

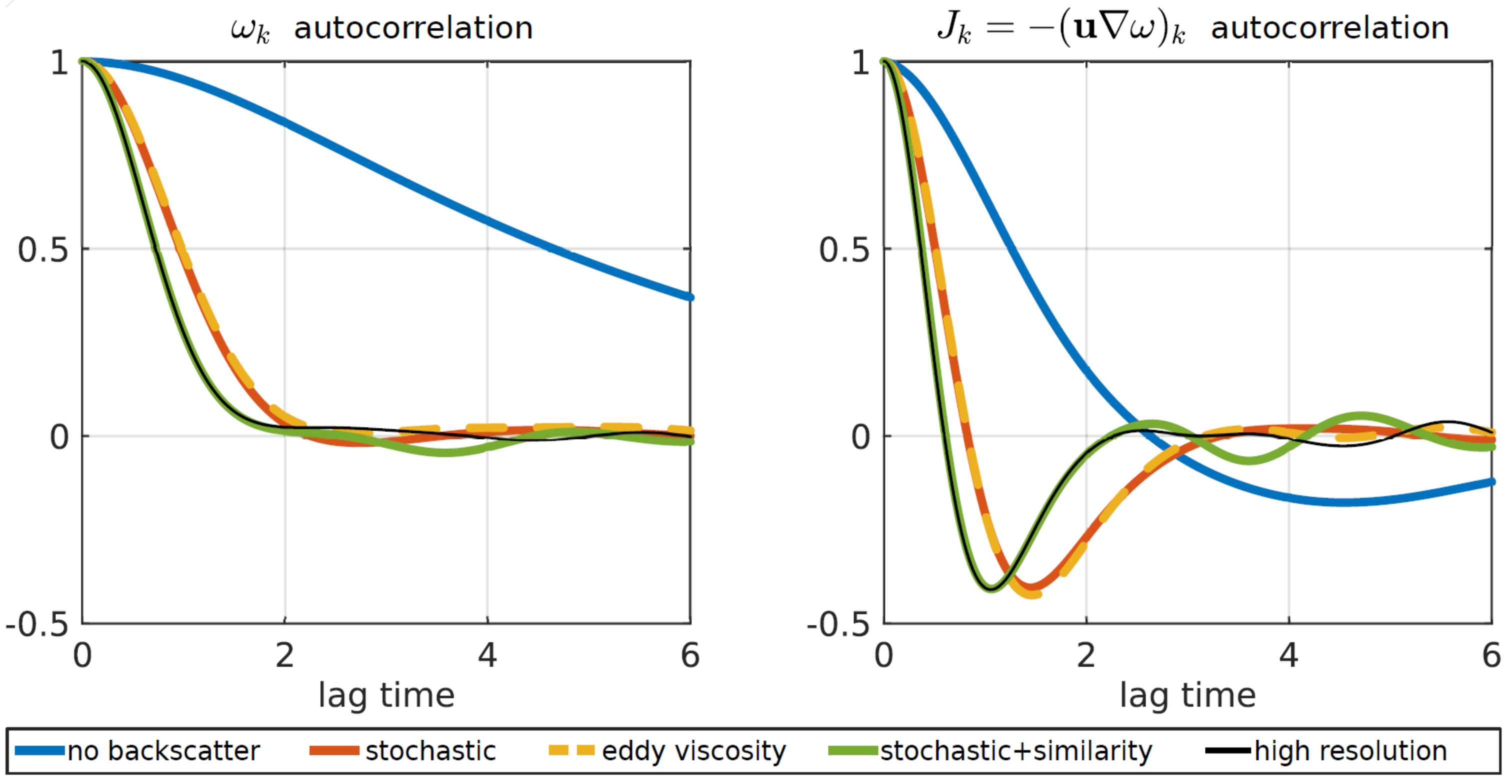
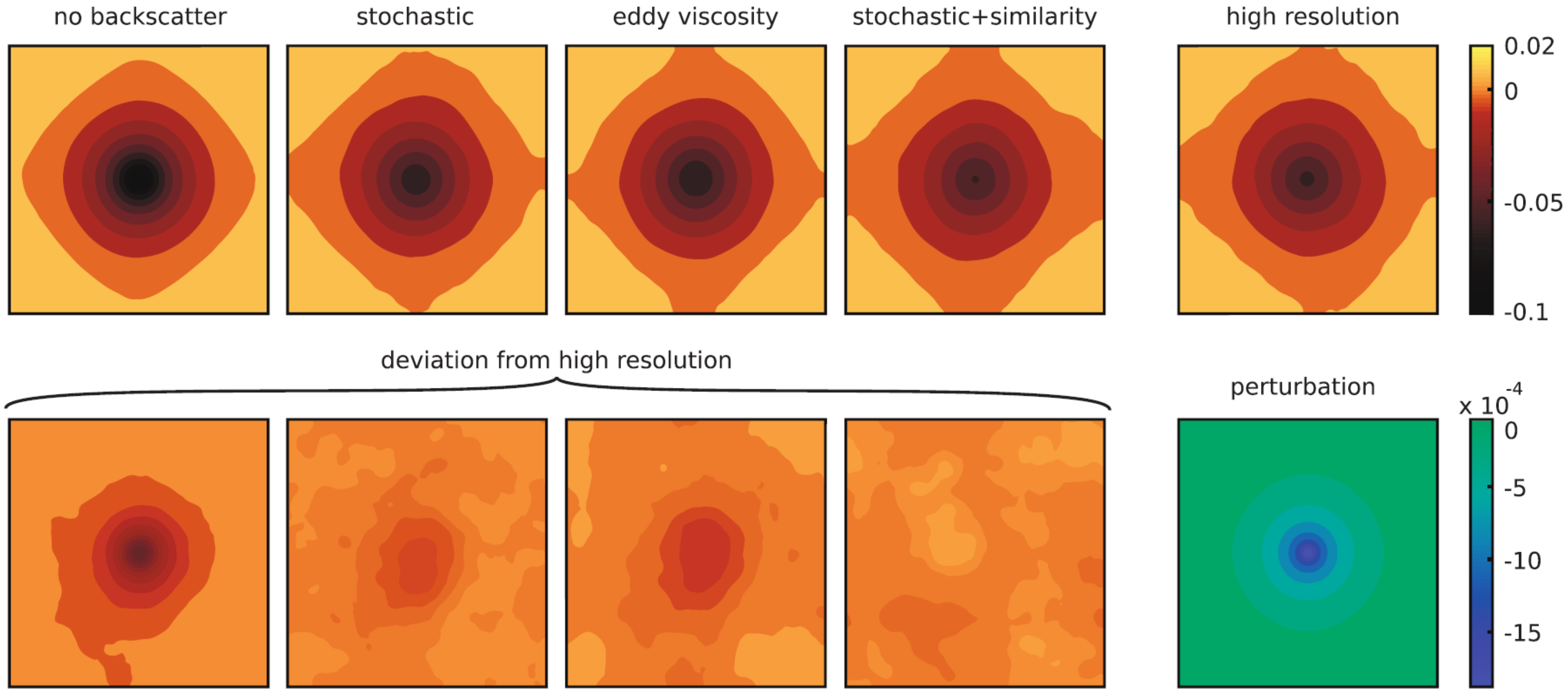


Fig. 4. Autocorrelation functions of Fourier coefficients for $k = 30$, scheme E.

- Autocorrelation functions of solution and advection rhs were restored

Experiments with coarse models - 4



- Time averaged response to the small constant perturbation (**sensitivity**) has true extreme values for combined model (shown results is for scheme E)

Experiments with coarse models -5

- Error of time averaged response reduces for 5-7 times in $\|\cdot\|_\infty$ norm and for 3-4 times in $\|\cdot\|_2$ norm

	$\|\|\langle\psi\rangle - \langle\psi_{dns}\rangle\|\|_\infty / \|\|\langle\psi_{dns}\rangle\|\|_\infty$				$\|\|\langle\psi\rangle - \langle\psi_{dns}\rangle\|\|_2 / \|\|\langle\psi_{dns}\rangle\|\|_2$			
scheme	E	INMCM	Z	CCS	E	INMCM	Z	CCS
no backscatter	0.83	0.47	0.68	0.58	0.44	0.30	0.39	0.37
stochastic	0.13	0.08	0.09	0.10	0.13	0.11	0.10	0.11
eddy viscosity	0.17	0.09	unstable	0.11	0.19	0.13	unstable	0.14
stochastic+similarity	0.07	0.08	0.05	0.05	0.11	0.10	0.07	0.08

Conclusions

- Parametrizations reproduce inverse energy cascade (energy spectrum, stream function patterns, autocorrelation functions)
- These improvements in dynamics are due to restoration of internal variability (parametrizations are small in norm compared to rhs)
- Stochastic and eddy viscosity parametrizations give almost the same results however average response for eddy viscosity model is quite worse. Also it could be unstable (scheme Z).
- **Combined model** (stochastic+similarity) **gives the best results** and demonstrates restoration of energy spectrum in middle and large scales at the same time. Also it has the best sensitivity among all the investigated parametrizations.