High-order approximation schemes for the staggered reduced lat-lon grid

Gordey Goyman, Vladimir Shashkin

Marchuk Institute of Numerical Mathematics RAS (INM RAS)

Hydrometeorological Center of Russia (HMCR)

Moscow Institute of Physics and Technology (MIPT)

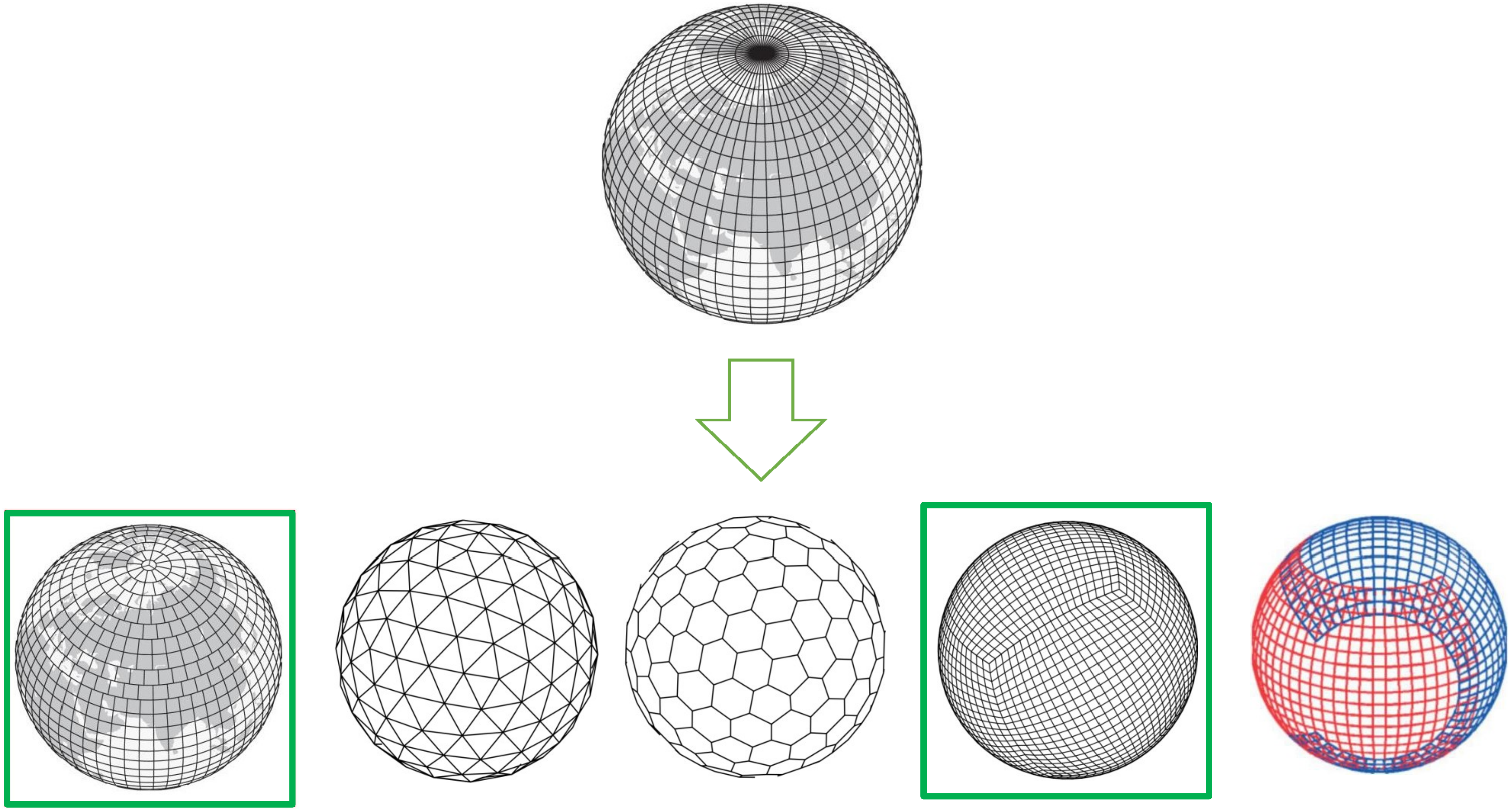
gordeygoyman@gmail.com

CITES 2021

Outline

- Motivation
- Staggered reduced grid schemes
- Numerical experiments
- Conclusion

Quasi-uniform horizontal grids



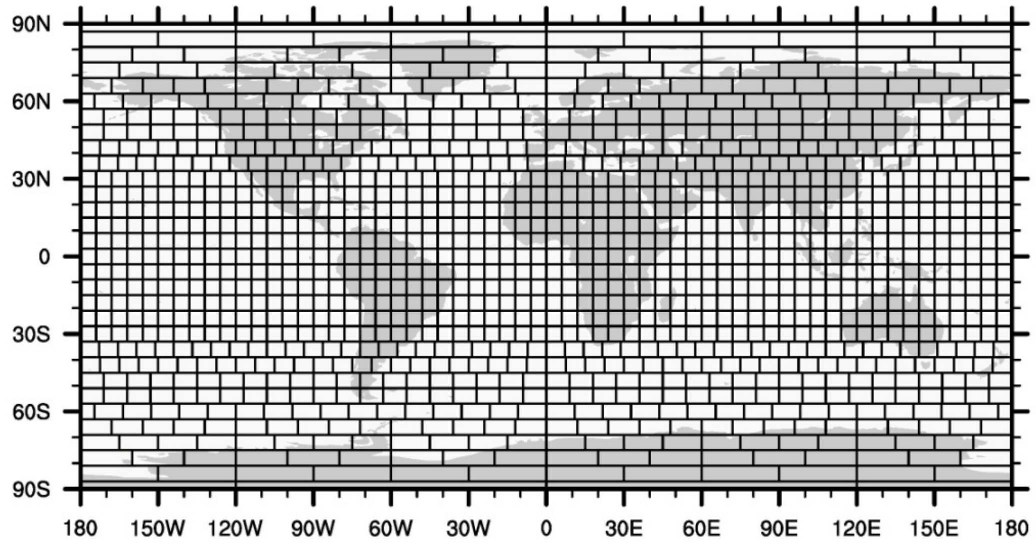
Reduced lat-lon grid

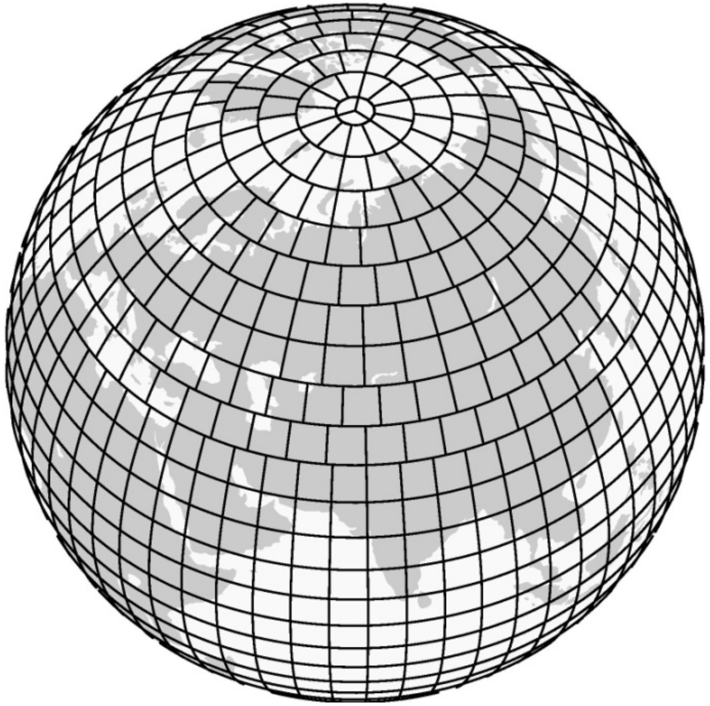
Pros

- Quasi-uniform
- Semi-structured
- Orthogonal curvilinear coordinates
- Geophysical flow alignment with grid lines

Successfully used in semi-Lagrangian models with spectral or Fourier transform based approximations (IFS, SL-AV, …)

Lack of convergence for standard FD, FE, FV methods. So called reduced grid pole problem (Shuman 1970)





Reduced grid pole problem

Number of points at the circumpolar latitude is fixed with the increase of grid resolution

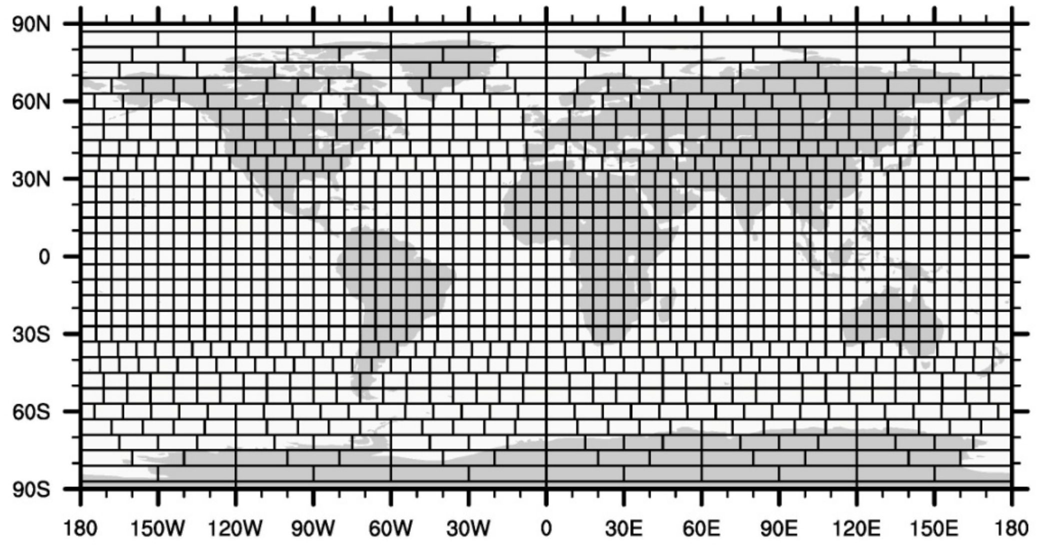
⬇

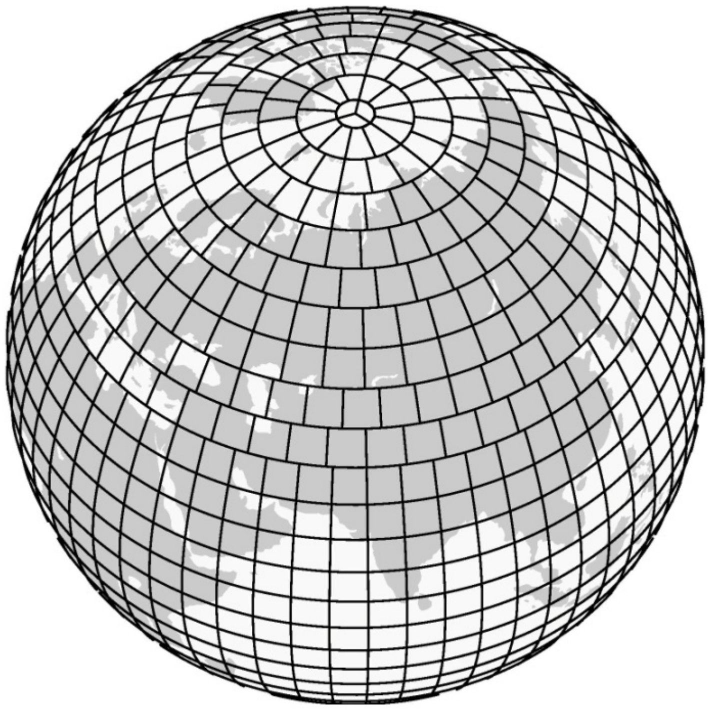
Discretization errors of longitudinal operators do not decrease or can be unbounded due to $\frac{1}{\cos\varphi}$ factor

P Bénard, M Glinton QJ2019 – analysis of reduced grid pole problem

Reguced grid problem can be avoided with the use of approximations exact for $\cos m\lambda, \sin m\lambda$ with $m \leq 2$

Explicit vector-invariant SWE model tests



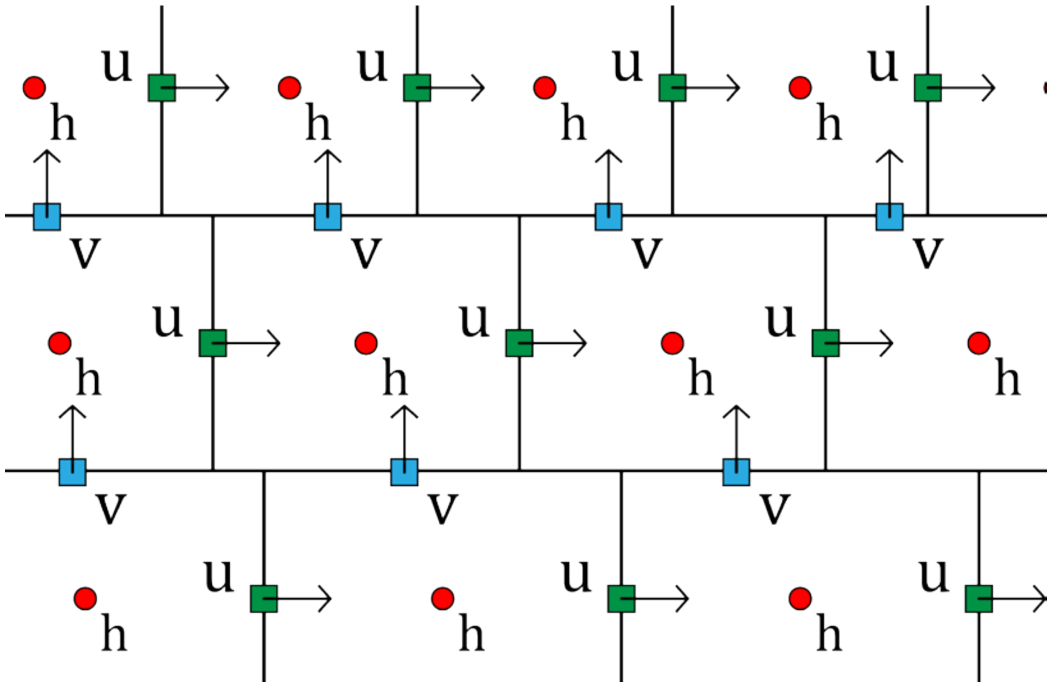


Shallow-water model

$$\begin{cases} \dfrac{D\boldsymbol{V}}{Dt} = -2\boldsymbol{\Omega} \times \boldsymbol{V} - \nabla(\phi + \phi_s) \\ \dfrac{D\phi}{Dt} = -\phi(\nabla \cdot \boldsymbol{V}) \end{cases}$$

- Semi-implicit semi-Lagrangian time-stepping
- Staggered grid approximations of differential operators with local stencils
- BiCGstab/multigrid solver of the Helmholtz problem

Reduced lat-lon grid C-staggering

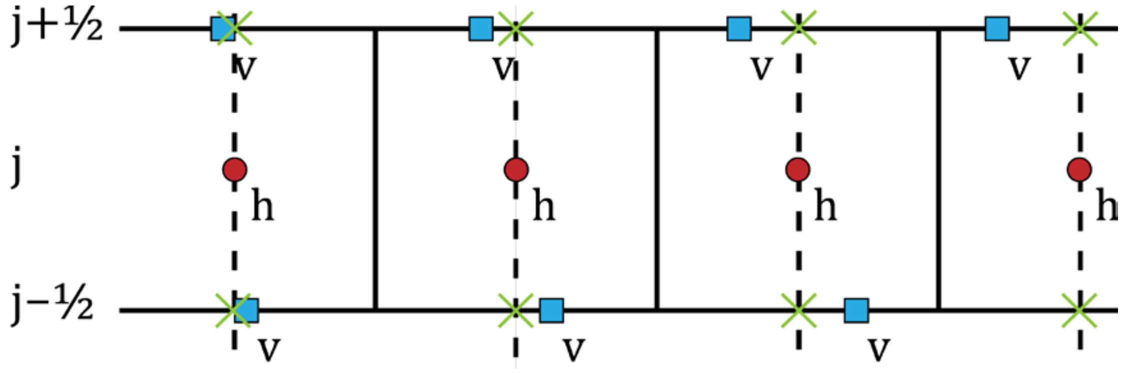


- h-points are located latitudes φ_j with lon-spacing $\Delta\lambda_j$
- u-points half-step shifted in lon-direction from the h-points
- v-points are located at the latitudes $\varphi_{j+1/2}$ with lon-spacing $\Delta\lambda_{j+1/2}$

$$N_\lambda^{j+1/2} = (N_\lambda^{j} + N_\lambda^{j+1})/2$$

Further details: Goyman G. S., Shashkin V. V. Horizontal approximation schemes for the staggered reduced latitude-longitude grid. JCP 2021

Differential operators approximation



Grid points are not aligned along meridional lines. Weighted sum of neighbor grid points values should be used

Regular grid

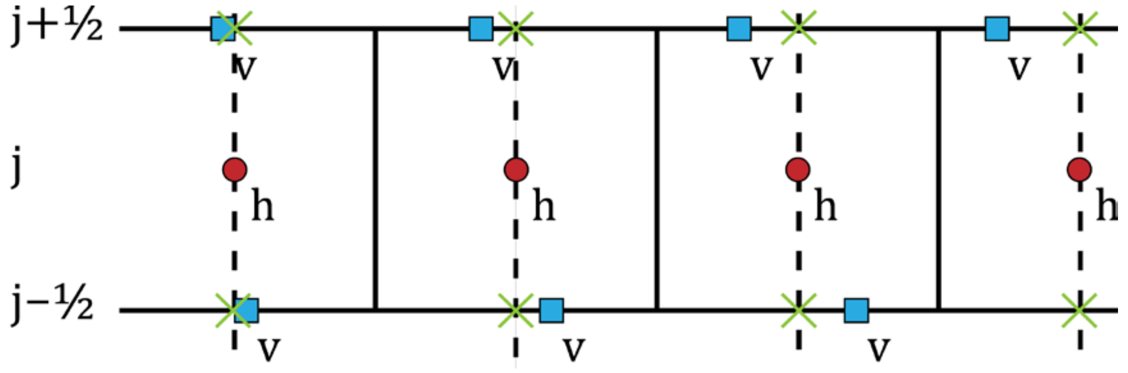
$$\left(\frac{\partial q}{\partial \varphi}\right)_{i,j+\frac{1}{2}} \approx \frac{q_{i,j+1} - q_{i,j}}{\Delta\varphi}$$

⇨

Reduced grid

$$\left(\frac{\partial q}{\partial \varphi}\right)_{i,j+\frac{1}{2}} \approx \frac{\sum_k \alpha_k q_{k,j+1} - \sum_p \beta_p q_{p,j}}{\Delta\varphi}$$

Differential operators approximation



Grid points are not aligned along meridional lines. Weighted sum of neighbor grid points values should be used

Regular grid

$$\left(\frac{\partial q}{\partial \varphi}\right)_{i,j+\frac{1}{2}} \approx \frac{q_{i,j+1} - q_{i,j}}{\Delta\varphi}$$

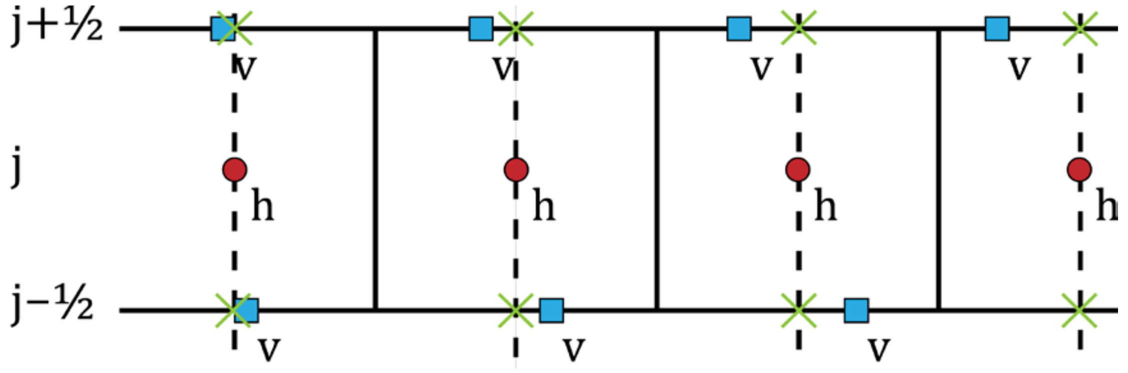
⇒

Reduced grid

$$\left(\frac{\partial q}{\partial \varphi}\right)_{i,j+\frac{1}{2}} \approx \frac{\sum_k \alpha_k q_{k,j+1} - \sum_p \beta_p q_{p,j}}{\Delta\varphi}$$

Denote $q_j = \left(q_{1,j}, q_{2,j}, \ldots, q_{N_j^{\lambda},j}\right)$ vector of grid function values at a fixed latitude

Differential operators approximation



Grid points are not aligned along meridional lines. Weighted sum of neighbor grid points values should be used

Regular grid

$$\left(\frac{\partial q}{\partial \varphi}\right)_{i,j+\frac{1}{2}} \approx \frac{q_{i,j+1} - q_{i,j}}{\Delta\varphi}$$

Reduced grid

$$\left(\frac{\partial q}{\partial \varphi}\right)_{i,j+\frac{1}{2}} \approx \frac{\sum_k \alpha_k q_{k,j+1} - \sum_p \beta_p q_{p,j}}{\Delta\varphi}$$

Denote $q_j = \left(q_{1,j}, q_{2,j}, \ldots, q_{N_j^\lambda,j}\right)$ vector of grid function values at a fixed latitude

$$\left(\frac{\partial q}{\partial \varphi}\right)_{j+\frac{1}{2}} \approx \frac{W_{j+\frac{1}{2}}^{j+1} q_{j+1} - W_{j+\frac{1}{2}}^{j} q_j}{\Delta\varphi}$$

$W_{j+\frac{1}{2}}^{j+1}, W_{j+\frac{1}{2}}^{j}$ Interpolation matrices from 1D grids at latitudes $j, j+1$ to grid at latitude $j+\frac{1}{2}$

Differential operators approximation

Starting from standard 4th order regular C-grid discretization

$$(\mathrm{div}_\varphi v)_j = \frac{27\left(W_j^{j+\frac{1}{2}}\tilde{v}_{j+\frac{1}{2}} - W_j^{j-\frac{1}{2}}\tilde{v}_{j-\frac{1}{2}}\right) - \left(W_j^{j+\frac{3}{2}}\tilde{v}_{j+\frac{3}{2}} - W_j^{j-\frac{3}{2}}\tilde{v}_{j-\frac{3}{2}}\right)}{24\cos\varphi_j \Delta\varphi_j}$$

$$\tilde{v}_{j+\frac{1}{2}} = (v\cos\varphi)_{j+\frac{1}{2}}$$

$$(\mathrm{grad}_\varphi h)_{j+\frac{1}{2}} = \frac{27\left(W_{j+\frac{1}{2}}^{j+1} h_{j+1} - W_{j+\frac{1}{2}}^{j} h_j\right) - \left(W_{j+\frac{1}{2}}^{j+2} h_{j+2} - W_{j+\frac{1}{2}}^{j-1} h_{j-1}\right)}{24\Delta\varphi_{j+\frac{1}{2}}}$$

Interpolation procedures – piecewise 5 point trigonometric interpolation (exact for $\cos m\lambda$, $\sin m\lambda$ with $m \leq 2$)

Differential operators approximation

4^{th} order longitude differentiation exact for $\cos m\lambda$, $\sin m\lambda$ with $m \le 2$

$$(\delta_\lambda q)_{i,j} = \frac{1}{4\sin\Delta\lambda_j(\cos\frac{\Delta\lambda_j}{2}+1)}\left((2\cos\frac{\Delta\lambda_j}{2}+1)^2(q_{i+\frac{1}{2},j}-q_{i-\frac{1}{2},j}) - \frac{1}{4\cos^2\frac{\Delta\lambda_j}{2}-1}(q_{i+\frac{3}{2},j}-q_{i-\frac{3}{2},j})\right)$$

$$(\mathrm{div}_\lambda u)_{i,j} = \frac{(\delta_\lambda u)_{i,j}}{\cos\Delta\varphi_j}$$

Coriolis terms

$$\begin{cases} \dfrac{Du}{Dt} - fv + \cdots = 0 \\ \dfrac{Dv}{Dt} + fu + \cdots = 0 \end{cases}$$

Piecewise cubic interpolation in φ-direction

Piecewise trigonometric interpolation in λ-direction

Semi-Lagrangian advection

Departure points trajectories – SETTLS scheme (Hortal 2002)

Departure points interpolation

Piecewise quantic interpolation in φ-direction
5 point piecewise trigonometric interpolation in λ-direction

Reduced grid construction

Number of points in longitude decreases polewards from $2N_\varphi$ to the value $N_p = \alpha 2N_\varphi$

$$N_j^\lambda = \left\lceil N_p + (2N_\varphi - N_p)\,\frac{\cos\varphi_j - \cos\varphi_1}{\cos\varphi_{N_\varphi/2} - \cos\varphi_1} \right\rceil,\ \ j = 1..\frac{N_\varphi}{2}.$$

Number of points along half-integer latitudes

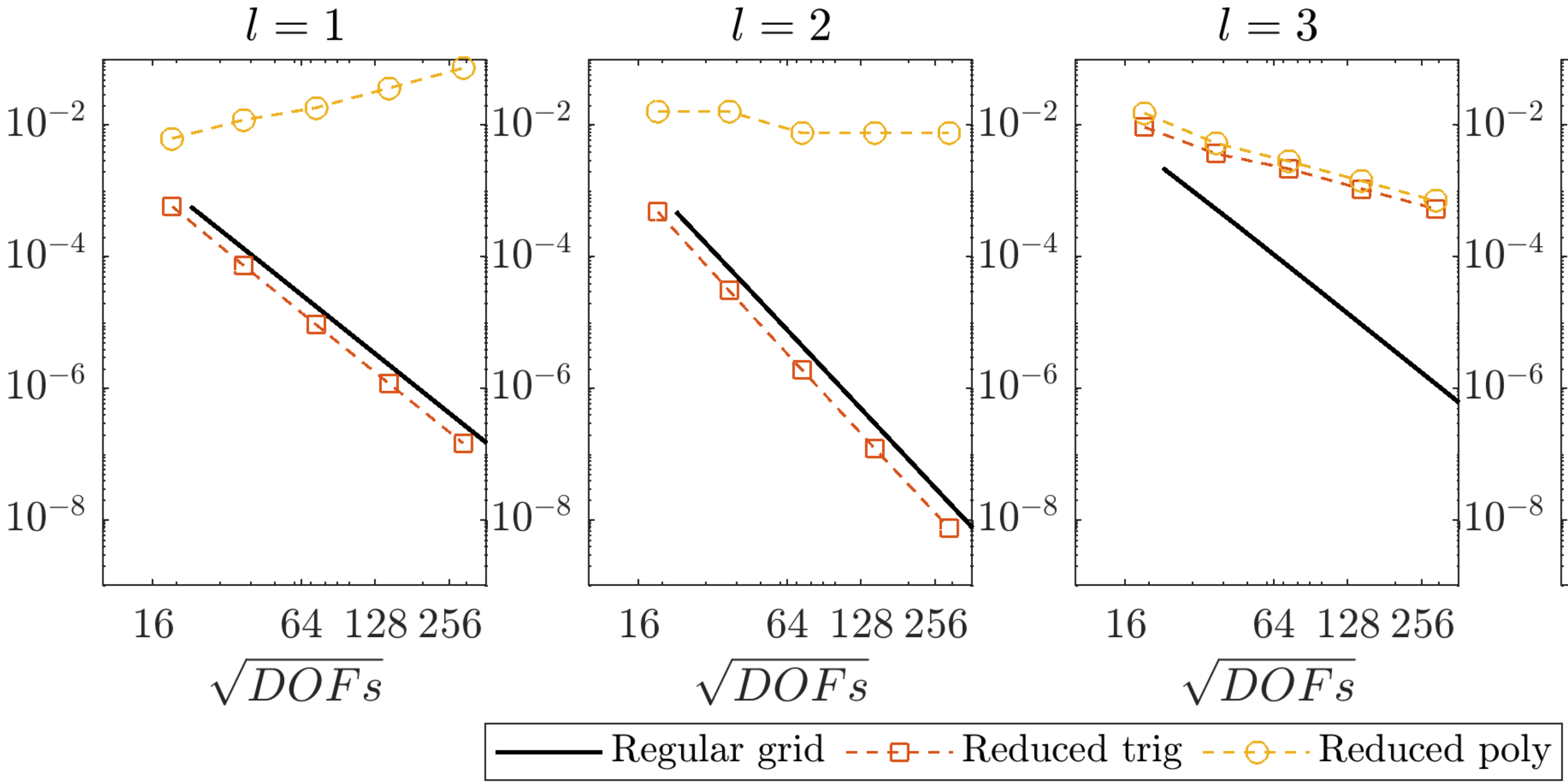
$$N_{j+1/2}^\lambda = \left\lceil \frac{N_{j+1}^\lambda + N_j^\lambda}{2} \right\rceil$$

Discrete Laplace operator convergence

Convergence of discrete Laplace operator

$$L = (\mathrm{div}_\lambda \mathrm{grad}_\lambda + \mathrm{div}_\varphi \mathrm{grad}_\varphi)$$

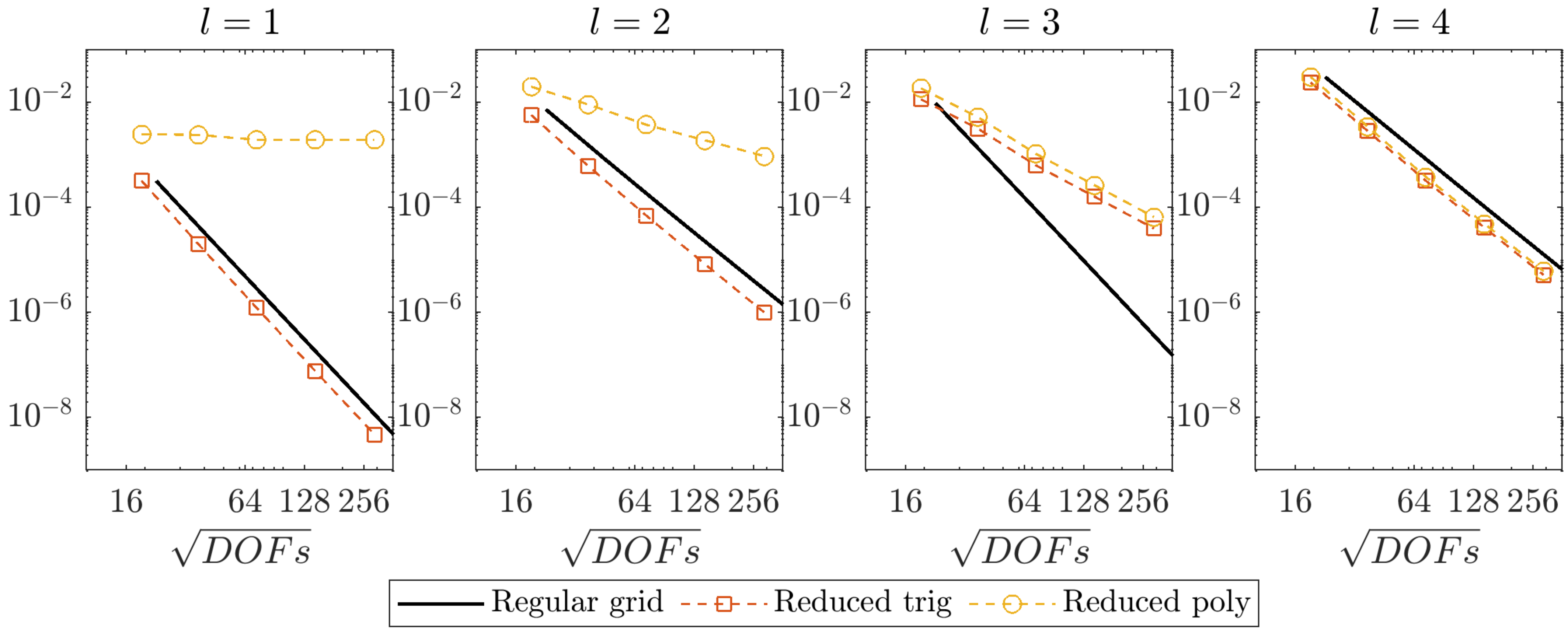
Test functions $Y_l^l = \cos^l \varphi \sin \lambda l$



Discrete Coriolis operator convergence

Convergence of discrete Coriolis operator

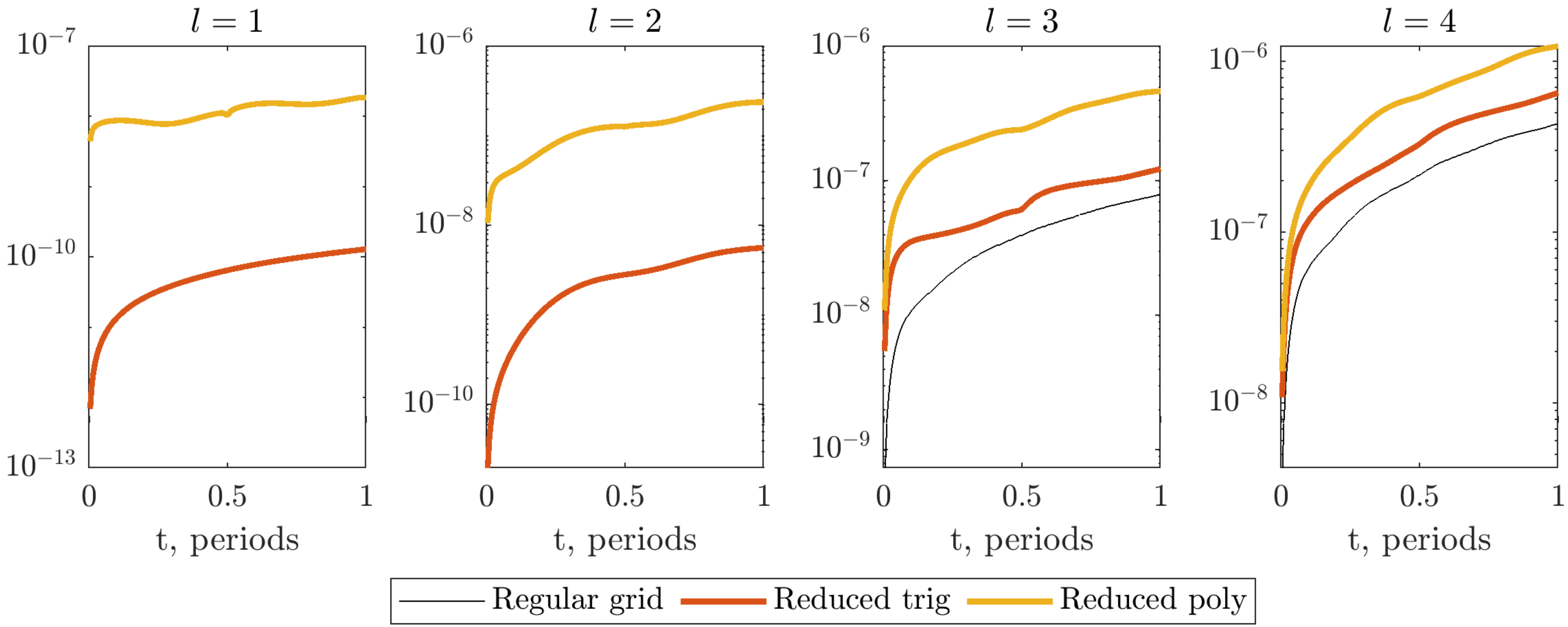
Test functions – vector spherical harmonics with $l = m$



Semi-Lagrangian advection

Advection of Y_l^l test function with solid body rotation wind field rotated by $\pi/4$

Time evolution of relative l_∞ error norm



Solid body rotation testcase

Geostrophically balanced flow with grid rotation $\gamma = \pi/4$

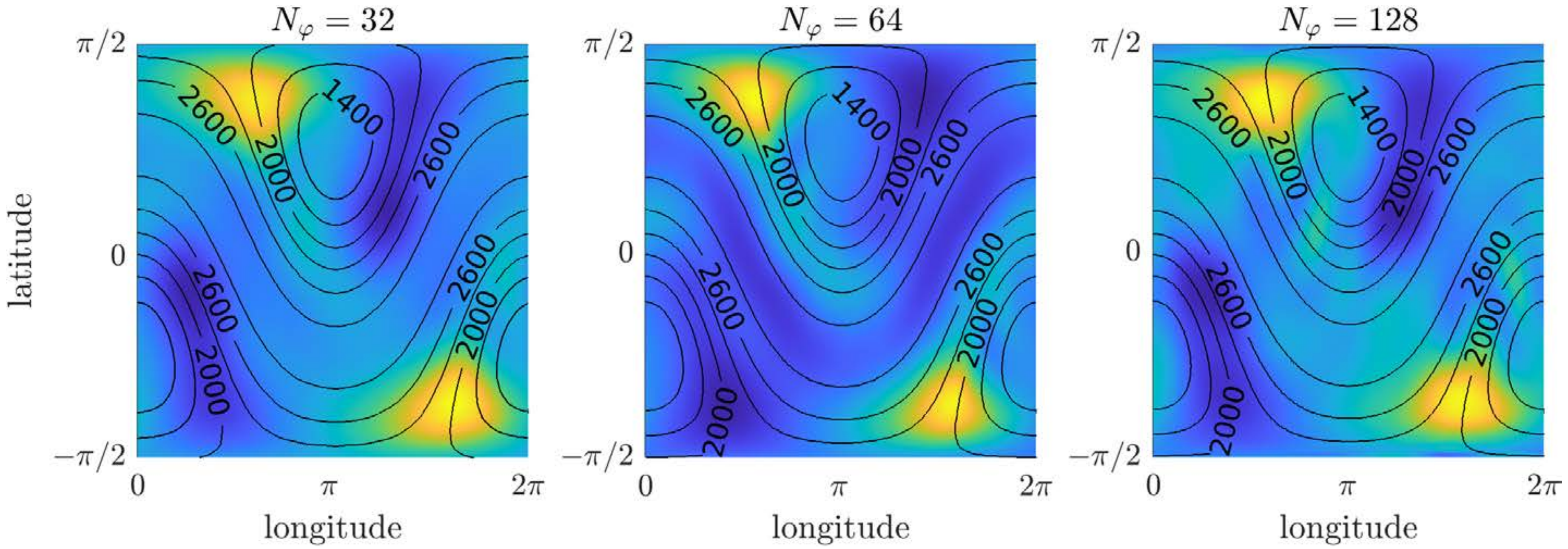
$$u = u_0 \left(\cos\varphi \cos\gamma + \cos\lambda \sin\varphi \sin\gamma\right),$$

$$v = -u_0 \sin\lambda \sin\gamma,$$

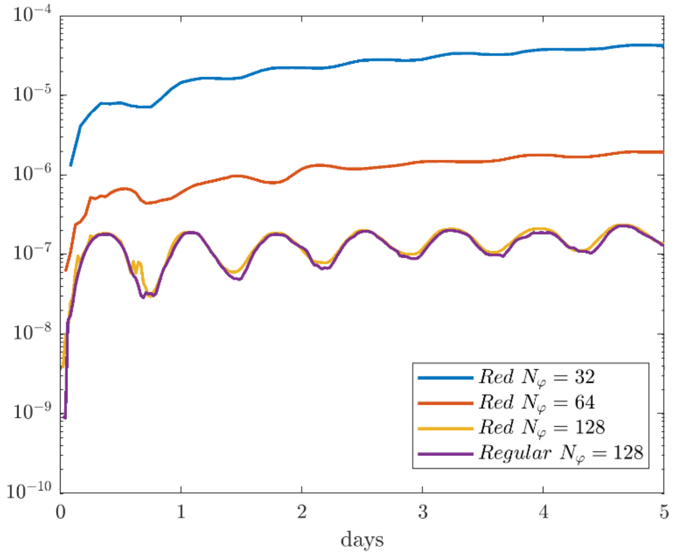
$$gh = gH - a\Omega u_o \left(-\cos\lambda \cos\varphi \sin\gamma + \sin\varphi \cos\gamma\right)^2.$$

Solid body rotation testcase

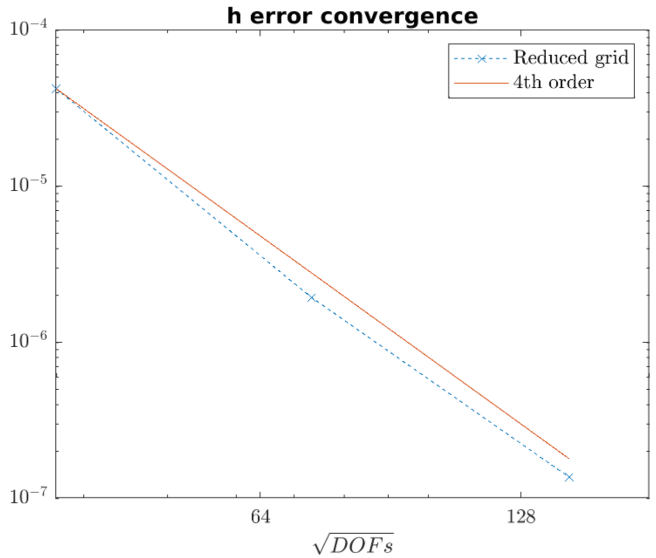
Поле h (контуры) и ошибка поля h (цвет) на 5 сутки модельного времени



Эволюция во времени нормы ошибки h в зависимости от разрешения сетки

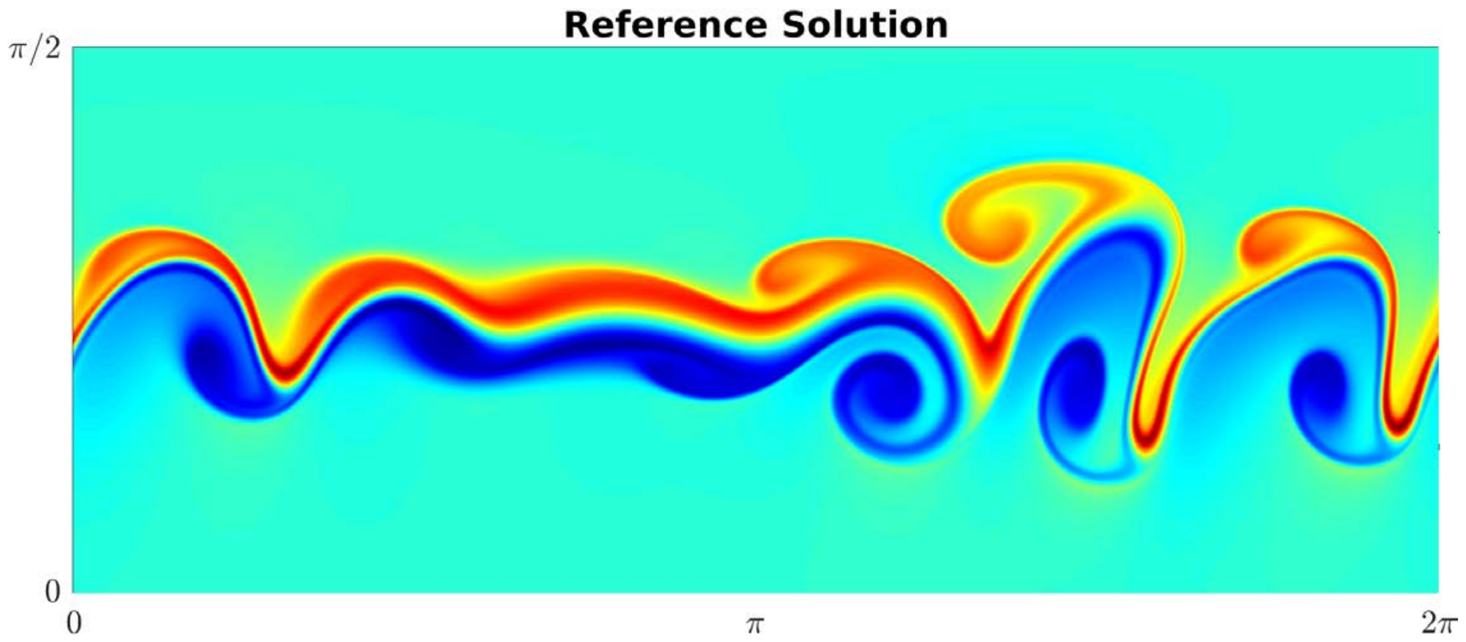


Сходимость нормы ошибки h в зависимости от разрешения сетки

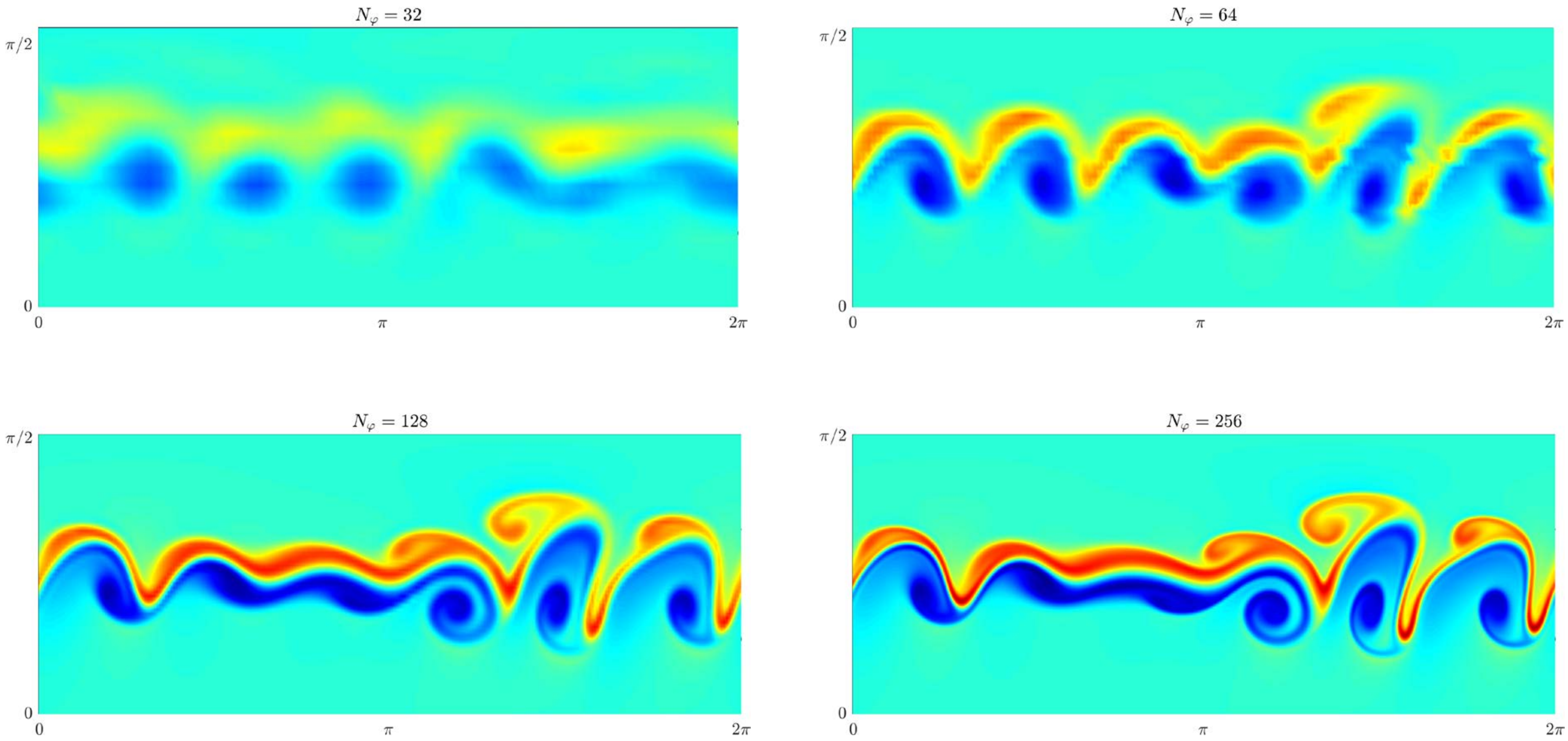


Barotropic instability test case

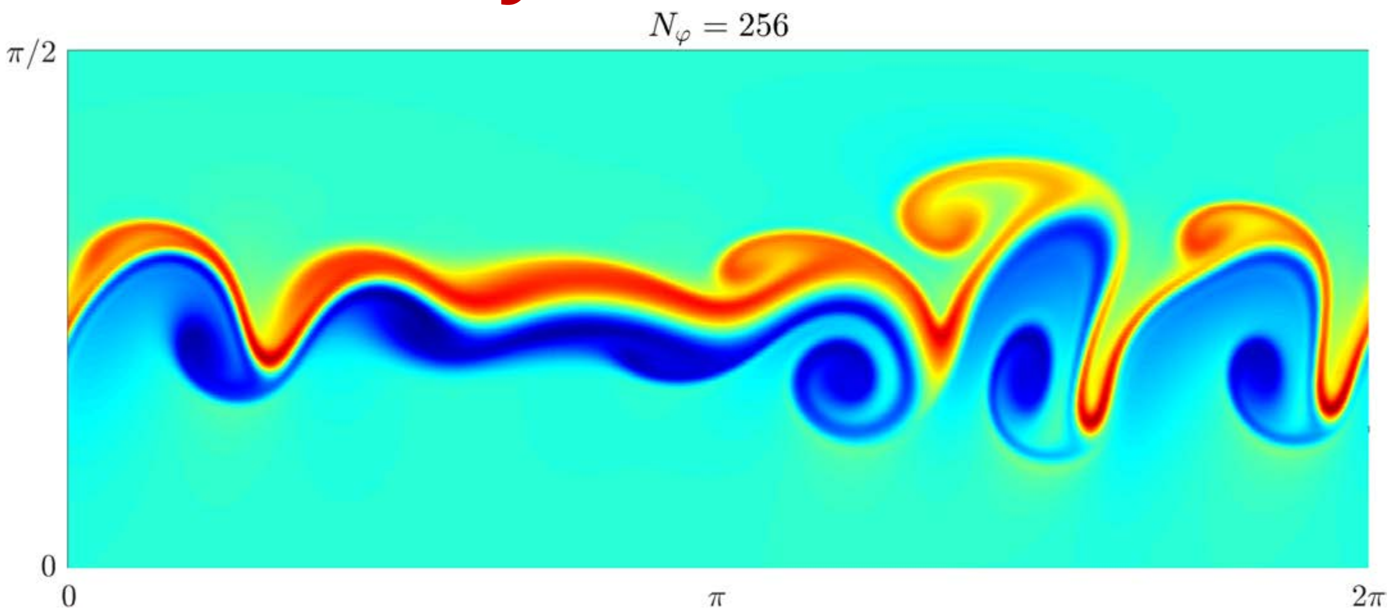
- balanced mid-latitude zonal flow with Gaussian perturbation in the height field
- Grid rotation $\gamma = \frac{\pi}{8}$
- Compare relative vorticity field after 6 days of simulation using model with different resolutions

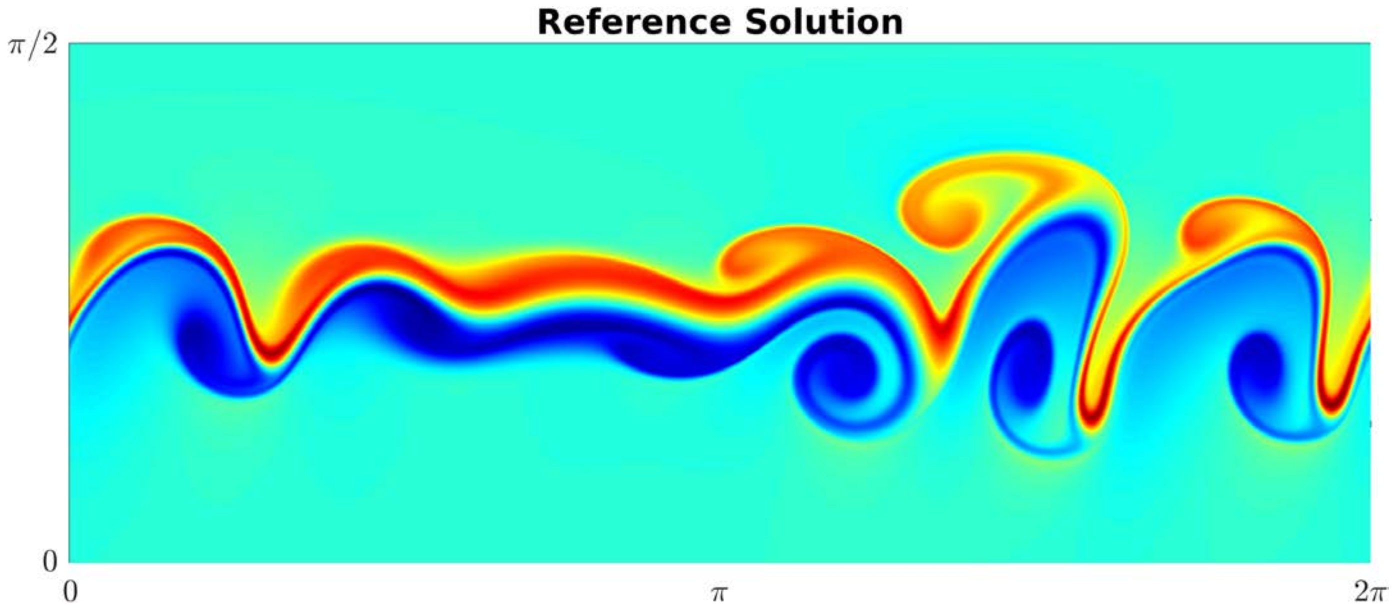


Barotropic instability test case



Barotropic instability test case





Summary

- Semi-implicit semi-Lagrangian shallow water model at the staggered reduced lat-lon grid is developed
- Finite difference 4^{th} order spatial discretization is proposed
- Good results within idealized testcases

Thank you for attention!

Barotropic instability test case